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CRANFIELD

HEAT TRANSFER THROUGH AN INCOMPRESSIBLE LAMINAR BOUNDARY LAYER

by

V. L. Shah

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Heat Transfer through an Incompressible Laminar Boundary Layer

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V. L. Shah, B.E.(Mech.), D.I.C., M.Sc.(Eng.).

SUMMARY

The study of the boundary layer theory has been done by reviewing briefly the boundary layer concept, the boundary layer equations, and the exact solutions for wedge flow.

All the methods for the calculations of the heat transfer coefficient are surveyed briefly, describing their procedure and their limitations.

CONTENTS

LIST OF SYMBOLS

x	Distance parallel to the wall and from the beginning of the boundary layer. ft.
x_o	Distance where the heating starts. ft.
y	Distance normal to the wall. ft.
r	Radius of the body. ft.
ξ	Variable of the integration
ψ	Stream function
δ	Boundary layer thickness. ft.
δ_1	Boundary layer displacement thickness. ft.
δ_2	Momentum thickness. ft.
Δ	Thermal boundary layer thickness. ft.
Δ_1	Thermal displacement thickness. ft.
Δ_2	Enthalpy flux thickness. ft.
Δ_4	Heat flux thickness. ft.
u	Velocity in the boundary layer parallel to wall. ft/sec.
u_1	Main stream velocity parallel to the wall. Ft/sec.
v	Velocity normal to the wall. ft/sec.
T	Temperature in the boundary layer. ^{o}F.
T_1	Main stream temperature. ^{o}F.
T_w	Temperature of the surface. ^{o}F.
θ	$\dfrac{T - T_w}{T_1 - T_w}$
μ	Viscosity. lb/ft.sec.
ν	Kinematic viscosity. ft^2/sec.
h	Heat transfer coefficient. B.T.U./sec.ft^2. F.
k	Thermal conductivity. B.T.U./sec.ft. F.

List of Symbols (Continued)

ρ	Density. lb/ft^3.
H	Enthalpy
C_p	Specific heat
α	Thermal diffusivity. $k/\rho C_p$
τ_ω	Shear stress at the surface. $lb/ft^3/sec^2$.
$\dot{q}_w''$	Heat transfer rate at the wall per unit area and unit time
-	Represents axisymmetric flow.
C_f	Friction factor. $\tau_\omega/\frac{1}{2}\rho u_1^2$
Nu_x	Nusselt Number. hx/k
Pr	Prandtl Number. $C_p\mu/k$
Re_x	Reynolds Number. U_1x/ν

1. Introduction

The calculation of the heat transfer to the laminar boundary layer is of importance in many engineering applications. This type of flow occurs in the aerodynamic heating of bodies in flight, in the cooling of gas turbine blades, along aerofoil surfaces, and along rocket motor nozzles. In such instances, the variation of the wall temperature and the variation of the main stream velocity profile have an important influence upon the rate of heat transfer, and a quantitative knowledge of heat transfer coefficient is necessary if the surfaces are deliberately heated or cooled.

Theoretically, the problem of heat transfer in a laminar boundary layer is to find, mathematically, a simultaneous solution to the three basic boundary layer equations for a given wall condition, longitudinal pressure distribution, and a gas with specified thermodynamic properties.

Analytical predictions of the local heat transfer, for flow over a flat plate and wedge, have met with considerable success. An explicit solution for more general problems of flow over a body of arbitrary shape, in which the velocity at the edge of the boundary layer varies arbitrarily, is either impossible, or in many cases so cumbersome and time-consuming, that it cannot be carried out in practice. It is, therefore, desirable to possess at least approximate methods of solution to be applied in cases where an exact solution is impossible, or cannot be obtained with the reasonable amount of work, even if the accuracy is only limited.

A considerable amount of research has been done in this field and numerous methods are available for calculating heat transfer over the bodies of arbitrary shape with arbitrary surface temperature. But there is not a single report available which covers all these methods describing their procedure and their limitations. This presents great difficulty to any beginner entering this field and also to an industrial designer. The purpose of this report is to survey all the literature pertaining to this field and to present their conclusions in an orderly manner.

In selecting suitable material from the extensive available literature, the basic objective has been to provide an introduction to the field of heat transfer through an incompressible laminar boundary layer. Thus the exact solution for wedge flow has been briefly reviewed after describing the basic boundary layer theory. All the approximate methods are then surveyed describing their procedure and the limitations.

Recently Spalding and Pun[(44)] have published a similar work in which they have surveyed all the methods very briefly and presented their conclusions in a tabular form.

2. Boundary Layer Theory

2.1. Introduction

In 1904, L. Prandtl proved that at a moderate Reynolds Number, the flow about a solid body can be divided into two regions.

(1) A very thin layer in the neighbourhood of the body called the boundary layer, where the vsicous terms and the conduction terms in the momentum and the energy equation play an essential part.

(2) The potential flow outside the boundary layer, where the viscous stresses are negligible compared to the inertia stresses.

This theory of Prandtl, with the few boundary layer assumptions, enabled the three basic equations of the boundary layer to be derived:

(1) Equation of Continuity
(2) Equation of Motion
(3) Energy Equation

Calculations of the heat transfer and the viscous stresses can be made by solving these three basic equations. The subject of the boundary layer theory can be divided into two main classes, depending on the type of flow:

1. Laminar
2. Turbulent

In laminar flow, the individual particles of fluid flow in a straight line parallel to the stream line without appreciable transverse to and fro motion, whereas in the turbulent flow innumerable eddies or vortices are present.

The laminar boundary layer can be further divided into three groups, depending on the velocity of the fluid:

1. High speed flow
2. Flow with moderate velocity
3. Natural convection

The present report is restricted to incompressible laminar flow with moderate velocity.

For incompressible flow, the presence of a boundary layer on a body influences the potential flow only in a secondary way, (unless reverse flow or separation occurs) through an alteration in the effective boundaries of the potential flow by the amount of the boundary layer displacement thickness. The potential flow, on the other hand, establishes the longitudinal pressure distribution for the boundary layer and thereby plays a controlling role in the behaviour and the formation of the boundary layer.

2.2. Boundary Layer Assumption

In the boundary layer theory, the velocity and temperature gradients parallel to the wall are assumed to be smaller than those in the direction normal to the wall.

This assumption leads to the following three assumptions:

1. Viscous shear stress and the heat conduction may be ignored in the direction parallel to the wall, in comparison with the viscous shear stress and the heat conduction in the direction normal to the wall.

2. The component of velocity normal to the wall is very small compared to the component parallel to the wall, and so the boundary layer flow is almost parallel to the wall.

3. The pressure within the boundary layer varies only in the direction parallel to the wall and, therefore, the pressure within the boundary layer is established by the potential flow outside the boundary layer.

These boundary layer assumptions are true when the boundary layer thickness is small compared to the radius of the curved body, and the distance from the stagnation point.

In the case of flow past a flat plate, the laminar boundary layer thickness is given by the equation

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}} \qquad (2.2.1)$$

Therefore, boundary layer assumptions do not hold good for low Reynolds Number and at the leading edge.

2.3. Dimensional analysis

An analytical solution of the boundary layer is very complex, due to the large number of terms involved. The method of dimensional analysis can be applied to simplify the solution. By this method the fundamental equations can be so arranged that the quantities enter the equations through certain combinations that are dimensionless. The forming of such equations is independent of the size of the units involved in the various terms in the equations. This method indicates the logical grouping of factors into dimensionless combinations. This is very helpful in interpreting data where two or more factors have been varied in different experiments.

The present report is restricted to the forced convection in a laminar incompressible flow with moderate velocities, where the heat due to friction and compression need not be taken into account. For such flows the dimensional analysis leads to the conclusion that the solution of the above system of equations, for the velocity field, temperature field, and local coefficient of heat transfer, depend upon the two dimensionless groups.

1. Reynolds Number, which is a ratio of the inertia force to the friction force.
2. Prandtl Number, which is a ratio of the kinematic diffusion to thermal diffusion.

Vectorial dimensional analysis further leads to a relationship:

$$\frac{Nu}{\sqrt{Re}} = f\,(Pr,\ Geometry) \qquad (2.3.1)$$

There is a third dimensionless group called Grashof Number which becomes important only at very small velocities of flow, particularly if the motion is carried out by the buoyancy forces and not by the pressure differences. In such cases the flow becomes independent of Reynolds Number and the process is called natural convection.

With moderate velocities, the buoyancy forces caused by temperature differences are small compared with the inertia and frictional forces. In such cases the problem ceases to depend on Grashof Number, and Nusselt Number depends only on Reynolds Number and Prandtl Number.

2.4. Boundary Layer Equations for two-dimensional flow

2.4.1. Differential equations

Three differential equations form the basis of the boundary layer theory and they are reviewed below.

(a) Continuity Equation

In the case of the steady flow, the law of the conservation of mass asserts that the mass flow entering the control volume is equal to the mass flow leaving the control volume.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (2.4.1)$$

and for incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.4.1)$$

(b) Momentum Equation

For steady flow, Newton's 2nd Law of Motion states that the increase in the momentum per unit time of all the particles passing through the control volume is equivalent to the inertia force, and must be in equilibrium with the external forces acting on the surface and within the control volume. The increase of the momentum of all the particles passing through the control volume can be expressed as the difference between the momentum leaving the volume per unit time, and the momentum entering through the surface per unit time. Thus

$$-\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho vu) \qquad (2.4.2)$$

Pressure force + Viscous force = Change in momentum

Expanding with the knowledge of the continuity equation, the equation of momentum simplifies to

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \rho u_1\frac{du_1}{dx} + \mu\frac{\partial^2 u}{\partial y^2} \qquad (2.4.3)$$

or

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \qquad (2.4.4)$$

where $u_1 \frac{du_1}{dx} = - \frac{1}{\rho} \frac{dp}{dx}$ (from the potential flow)

(c) Energy Equation

The law of the conservation of energy for the steady flow through a control volume states that the net efflux of enthalpy plus the kinetic energy is equal to the net rate of heat transfer into the control volume, plus the net rate of shear work into the control volume.

For moderate velocities the rate of shear work can be neglected. Therefore the energy equation is

$$\frac{\partial}{\partial x} (\rho u) \left[H + \frac{u^2 + v^2}{2} \right] + \frac{\partial}{\partial y} (\rho v) \left[H + \frac{u^2 + v^2}{2} \right] = \frac{\partial}{\partial y} \left[k \frac{\partial T}{\partial y} \right] \qquad (2.4.5)$$

Expansion of these terms, with the known continuity equation, simplifies equation 2.4.5 to

$$\rho u \frac{\partial}{\partial x} \left[H + \frac{u^2 + v^2}{2} \right] + \rho v \frac{\partial}{\partial y} \left[H + \frac{u^2 + v^2}{2} \right] = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \qquad (2.4.6)$$

Assumption that the enthalpy is a function only of temperature and the use of gas laws $\frac{p}{\rho} = RT$ leads to the thermodynamic relationship

$$dH = C_p \, dT \qquad \text{and} \qquad \frac{\partial H}{\partial x} = C_p \frac{\partial T}{\partial x}$$

The energy equation may now be written as

$$\rho u \left[C_p \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left(\frac{u^2 + v^2}{2} \right) \right] + \rho v \left[C_p \frac{\partial T}{\partial y} + \frac{\partial}{\partial y} \left(\frac{u^2 + v^2}{2} \right) \right]$$

$$= \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \qquad (2.4.7)$$

and, for moderate velocities, the kinetic energy is negligible. Assuming k : constant, the energy equation simplifies to

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{dy^2} \qquad (2.4.8)$$

Mathematically, the problem of the incompressible laminar boundary layer with moderate velocity may be summarised as that of finding the simultaneous solution to

(1) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.4.1)$$

(2) $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \qquad (2.4.4)$$

(3) $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (2.4.8)$$

for a given wall condition, longitudinal pressure distribution, and a gas with specified thermodynamic properties.

2.4.2. Integral Equations

Integral equations of motion and energy are obtained by integrating the differential equations of the boundary layer flow over the boundary layer thickness. These equations, rather than satisfying the boundary conditions for every individual fluid particle, satisfy the boundary layer flow only in the average.

These equations with assumed velocity and temperature profiles, form the basis of all the approximate methods.

1. Momentum equation

$$\frac{\tau_\omega}{\rho} = \frac{d}{dx}\left(u_1^2\,\delta_2\right) + \delta_1 u_1 \frac{du_1}{dx} \qquad (2.4.9)$$

where δ_1 and δ_2 are the displacement thickness and the momentum thickness respectively.

and

2. Heat flux equation

$$\dot{q}''_w = \frac{d}{dx}\int_0^\infty \rho\, C_p\, u\,(T - T_1)\, dy \qquad (2.5.0)$$

and the dimensionless form

$$St = \frac{d}{dx}\int_0^\Delta \frac{u}{u_1}\,(1 - \theta)\, dy \qquad (2.5.0)$$

The differential equations for the velocity (Equation of motion) and the thermal boundary layer (Energy equation) are very similar in structure. Except for the natural convection, the velocity field does not depend on the temperature field, although the converse is true.

At moderate velocities when the heat due to friction and compression may be neglected, the dependence of the temperature field on the velocity field is governed solely by the Prandtl Number. To each single velocity field there corresponds a single infinite family of temperature distribution, with the Prandtl Number as its parameter.

Exact investigation for wedge flow has been achieved with success and the results of wedge flow form the basis of comparison for all the approximate methods. In the next chapter the solution for the wedge is briefly reviewed.

2.5. Axisymmetric flow

The boundary layer, which exists on a cylindrical body (two dimensional), depends only on the potential flows around the cylinder. The shape of the cylindrical body does not enter the calculation explicitly, but indirectly influences the potential flow. On the other hand, in the axially symmetric flow, the boundary layer depends directly on the shape of the body, because the radius of the cross section appears explicitly in the differential equations, apart from influencing the potential velocity distribution.

Boundary layer equations for a fluid past an axisymmetric body can be derived in the same way as for the two dimensional flow.

They are

1. $$\frac{\partial(\bar{u}\,r)}{\partial\bar{x}} + \frac{\partial(\bar{v}\,r)}{\partial\bar{y}} = 0 \qquad (2.5.1)$$

(Continuity equation)

2. $$\bar{u}\,\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\,\frac{\partial\bar{u}}{\partial\bar{y}} = -\frac{1}{\rho}\,\frac{d\bar{p}}{d\bar{x}} + \nu\,\frac{\partial^2\bar{u}}{\partial\bar{y}^2} \qquad (2.5.2)$$

(Equation of motion)

3. $$\bar{u}\,\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{v}\,\frac{\partial\bar{T}}{\partial\bar{y}} = \alpha\,\frac{\partial^2\bar{T}}{\partial\bar{y}^2} \qquad (2.5.3)$$

(Energy equation)

Integral equations of motion and energy are obtained by integrating equations 2.5.2 and 2.5.3 respectively, over the boundary layer thickness.

Thus

$$\frac{\bar{\tau}_\omega}{\rho} = \frac{d}{d\bar{x}}\,(\bar{u}_1^2\;\bar{\delta}_2) + \bar{\delta}_1\;\bar{u}_1\,\frac{d\bar{u}_1}{d\bar{x}} + \bar{u}_1^2\,\frac{\bar{\delta}_2}{r}\,\frac{dr}{d\bar{x}} \qquad (2.5.4)$$

(Equation of motion)

$$\frac{1}{r}\,\frac{dr}{d\bar{x}}\int_0^{\bar{\delta}}\bar{u}\;\bar{T}\;d\bar{y} + \frac{d}{d\bar{x}}\int_0^{\bar{\delta}}\bar{u}\;\bar{T}\,d\bar{y} = -\alpha\left(\frac{\partial\bar{T}}{d\bar{y}}\right)_{\bar{y}=0} \qquad (2.5.5)$$

(Energy equation)

W. Mangler[8] has shown that, by using the transformations presented below, it is possible to transform these three boundary layer differential equations for an axisymmetr ic flow into the three differential equations for the two-dimensional case. Hence, it permits the use of the solutions for the two dimensional case to derive the solutions for axisymmetric flow.

According to Mangler, the equations which transform the co-ordinates, velocities, and thermal properties of the axisymmetrical problem to those of the equivalent two-dimensional are as follows

1. $$x = \frac{1}{L^2} \int_0^{\bar{x}} r^2(\bar{x})\, d\bar{x}$$

2. $$y = \frac{r(\bar{x})}{L} \bar{y}$$

3. $$u = \bar{u}$$

4. $$u_1 = \bar{u}_1$$

5. $$v = \frac{L}{r(\bar{x})} \left[\bar{v} + \frac{r'}{r} \bar{y}\, \bar{u} \right]$$

6. $$h = \frac{L}{r} \bar{h}$$

7. $$\Delta = \frac{r}{L} \bar{\Delta}$$

With the above-mentioned transformations it is possible to apply a solution of two-dimensional case to an axisymmetrical case. Hence, in the present report, attention is given to the two-dimensional flow and wherever necessary a comparison with an axisymmetrical case is made.

3. Wedge Flow

3.1. Velocity Boundary Layer

3.1.1. Momentum equation

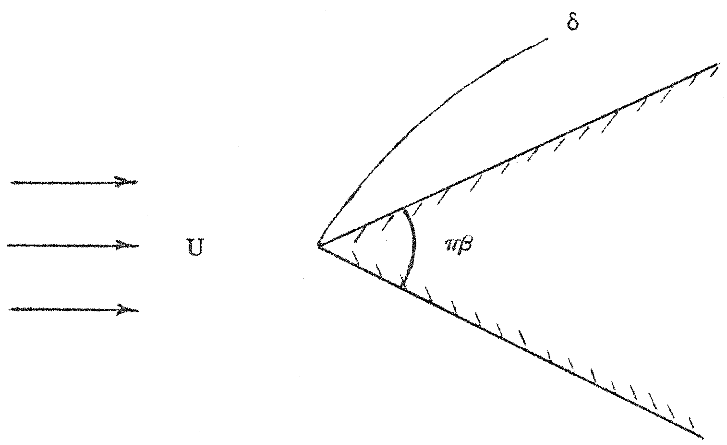

FIG. 3.0. FLOW AROUND A WEDGE

In the case of the flow past a wedge, the velocity of the potential flow is proportional to a power of the length co-ordinate from the stagnation point. i.e. $u_1 = c\,x^m$

where 1. $m = \dfrac{\beta}{2 - \beta}$

and 2. $\pi\beta$: angle between the two planes of the wedge.

This condition reduces the partial differential equation of motion to an ordinary differential equation. Equation of motion is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1\frac{du_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (3.1.1.)$$

with the transformations

$$f = \sqrt{\frac{m+1}{2}}\,\frac{\psi}{\sqrt{u_1 \nu x}} \quad \text{and} \quad \xi = \sqrt{\frac{m+1}{2}}\sqrt{\frac{u_1 x}{\nu}}\,y \qquad (3.1.2)$$

the equation of motion reduces to

$$f''' + f f'' + \frac{2m}{m+1}\left[1 - f'^2\right] = 0 \qquad (3.1.3)$$

where the boundary conditions are

$$\left.\begin{array}{l} \xi = 0 \;;\; f = 0,\; f' = 0 \\ \xi = \infty \;;\; f' = 1 \end{array}\right\} \qquad (3.1.4)$$

It is interesting to note that for the case $\beta : \frac{1}{2}$, $m : \frac{1}{3}$, the equation 3.1.3 becomes

$$f''' + f f'' + \tfrac{1}{2}(1 - f'^2) = 0 \qquad (3.1.5)$$

The differential equation is identical to the differential equation of rotationally symmetrical flow with stagnation point. Therefore, the solution for a two-dimensional wedge flow with $\beta : \frac{1}{2}$ is also a solution for the axially symmetrical flow with stagnation point.

Equation 3.1.3 was first deduced by Falkner and Skan[24] and its solution was later investigated in detail by Hartree[11]. The solution in terms of the velocity profile is represented in Fig. 3.1.

3.1.2. Flat Plates

For a flat plate, where $m : 0$, the equation 3.3 simplifies to

$$f''' + f f'' = 0 \qquad (3.1.6)$$

This equation is the Blasius non-linear differential equation and of the third order. The three boundary conditions are, therefore, sufficient to determine the solution completely.

Blasius[32] solved this equation in the form of a power series but Howarth[14] has solved this equation with a higher degree of accuracy. Velocity profile obtained by Howarth is shown in Fig. 3.2.

Later, Piercy and Preston[33] pointed out another method which results in a simple solution by a successive approximation in the equation.

$$f' = \frac{\int_0^{\xi} e^{-\int f\,d\xi}\,d\xi}{\int_0^{\infty} e^{-\int f\,d\xi}\,d\xi} \qquad (3.1.7)$$

3.1.3. Shear Stress

Shear stress τ_ω on the wall at the position x is given by

$$\tau_\omega = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (3.1.8)$$

i.e.

$$\frac{C_f}{2} = \frac{\tau_\omega}{\rho u_1^2} = \frac{\sqrt{\frac{m+1}{2}}\;\left[f''\right]_{\xi=0}}{\sqrt{\frac{u_1 x}{\nu}}} \qquad (3.1.9)$$

Values of $\left[f''\right]_{\xi=0}$ for some values of m are reproduced in Table 3.1 and the velocity distribution is shown in Fig. 3.1.

m	β	$\left[f''\right]_{\xi=0}$	$C_f \times \sqrt{Re_x}$
- 0.091	- 0.199	0	0
- 0.0476	- 0.10	0.319	0.44
0	0	0.470	0.664
0.111	0.2	0.687	1.024
0.333	0.5	0.928	1.516
1	1	1.233	2.466
∞	2	1.687	∞

TABLE 3.1.

3.2. Thermal Boundary Layer

3.2.1. Energy Equation

For incompressible flow, in which the dissipation is neglected, the energy equation is

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (3.2.1)$$

Equation 3.2.1 is linear in temperature, and consequently any sum of solutions of equation 3.2.1. is also a solution of equation 3.2.1. Equation 3.2.1 is transformed by utilising the variables of the Hartree equation (eq. 3.1.3) and a temperature function $(T_w - T_1) = B'x^r$. For the particular case of this power function wall temperature variation, an ordinary differential equation is obtained. This is

$$\theta'' + f \Pr \theta' - \Pr \left(\frac{2}{m+1}\right) r f'\theta = 0 \qquad (3.2.2)$$

where the boundary conditions are

$$\left.\begin{aligned} \theta &= 0 \quad \text{for} \quad \xi = 0 \\ \text{and} \qquad \theta &= 1 \quad \text{for} \quad \xi = \infty \end{aligned}\right\} \qquad (3.2.3)$$

3.2.2. r : 0

For the isothermal case, r : 0, the equation 3.2.2 simplifies to

$$\theta'' + f \Pr \theta' = 0 \qquad (3.2.4)$$

and the solution is

$$\theta = \frac{\int_0^{\xi} e^{-\Pr \int f \, d\xi} \, d\xi}{\int_0^{\infty} e^{-\Pr \int f \, d\xi} \, d\xi} \qquad (3.2.5)$$

Pohlhausen[13] has solved this equation for a flat plate and a few values of Prandtl Number. Eckert has solved the equation for a Prandtl Number of 0.7 and for a few values of m.

It can be seen from equations 3.2.5 and 3.1.2 that for the flow over a flat plate with Prandtl Number equal to unity:

1. The velocity distribution and the temperature distribution in the boundary layer are identical.

2. The thickness of the boundary layers, both the thermal and the hydrodynamic, are equal.

3. $St = \frac{h}{\rho C_p u_1} = \frac{\tau_\omega}{\rho u_1^2} = \frac{C_f}{2}$ (3.2.6)

Temperature distribution for the flat plate in the boundary layer, calculated by Pohlhausen, is shown in Fig. 3.3. As already mentioned, the curve for Prandtl Number : 1 gives also the velocity distribution in the boundary layer.

3.2.3. Heat transfer coefficient

Heat transfer coefficient is given by

$$h = -k \left(\frac{\partial \theta}{\partial y}\right)_{y=0} \tag{3.2.7}$$

or

$$\frac{Nu}{\sqrt{Re}} = \sqrt{\frac{m+1}{2}} \; (\theta')_{\xi=0} = f\,(Pr) \tag{3.2.8}$$

where

$$(\theta')_{\xi=0} = \frac{1}{\int_0^\infty e^{-Pr\int f\,d\xi}\,d\xi} \tag{3.2.9}$$

Results for some values of m and Prandtl Number are reproduced in Tables 3.2 and 3.3.

Pr	f(Pr)	Pr	f(Pr)	Pr	f(Pr)
0.6	0.276	0.9	0.320	7	0.645
0.7	0.293	1	0.332	10	0.730
0.8	0.307	1.1	0.344	15	0.835

TABLE 3.2. m : 0

m	$[\theta']_{\xi=0}$	$\frac{Nu}{\sqrt{Re}}$
0	0.414	0.2925
0.111	0.444	0.331
0.333	0.471	0.384
1	0.496	0.496
4	0.514	0.813

TABLE 3.3. Pr : 0.7

A good interpolation formula for m : 1 (Stagnation flow) is

$$\frac{Nu}{\sqrt{Re}} = 0.57\ Pr^{0.4} \tag{3.3.0}$$

and for m : 0 (flat plate)

$$\frac{Nu}{\sqrt{Re}} = 0.332\ Pr^{\frac{1}{3}} \tag{3.3.1}$$

For a fluid with a very large Prandtl Number, flowing over a flat plate, assumption of linear velocity profile leads to

$$\frac{Nu}{\sqrt{Re}} = 0.3387\ Pr^{\frac{1}{3}} \tag{3.3.2}$$

and for a fluid with a very low Prandtl Number, flowing over a flat plate, assumption of uniform velocity profile leads to

$$\frac{Nu}{\sqrt{Re}} = 0.565\ Pr^{\frac{1}{2}} \tag{3.3.3}$$

3.2.3. $(2-\beta)\ r : -1$

For this particular case, the equation 3.2.2 is exact and the solution by quadrature is possible.

Solution is

$$\theta = e^{-\int_0^{\xi} Pr\ f\ d\xi} \tag{3.3.4}$$

The temperature gradient at the wall

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = B'(\theta)_{\xi=0} \times \frac{1}{x\sqrt{\frac{2r}{m+1}}} \tag{3.3.5}$$

is seen to be zero for all values, except the indeterminate point at x : 0. At that point, the temperature of the fluid adjacent to wall is infinite and the heat is introduced impulsively at this point.

In the physical aspect, this is a case of an adiabatic wall when the heat is introduced impulsively at the stagnation point.

It can be seen from equation 3.3.5 that for $r\left(\frac{2}{m+1}\right) > -1$ the heat flow at the surface is from the surface to the boundary layer, and for $r\left(\frac{2}{m+1}\right) < -1$ the heat flow at the surface is from the boundary layer to the surface.

3.2.4. Other values of r

For other values of r, the numerical solution of equation 3.2.2 is necessary. Chapman and Rubesin[34], Schuh[22] and Levy[23] have carried out numerical solutions for certain values of r, Pr, and m. Levy has covered a wider range of r, Pr and m.

The results for Pr : 0.7 are plotted in Fig. 3.4.

From his results, Levy found that the local heat transfer coefficient can, with the exception of large negative r values, be expressed within 5 per cent as:

$$\frac{Nu}{\sqrt{Re}} = B(m, r)(Pr)^{\lambda} \qquad (3.3.6)$$

where the function B(m, r) can be approximated by the equation

$$B(m, r) = \left[\frac{2r}{m+1} + 1\right]^{0.37 + (\frac{0.12m}{m+1})} \times \left[0.57 \left(\frac{2m}{m+1} + 0.205\right)^{0.104}\right] \qquad (3.3.7)$$

and the exponent λ of the Prandtl Number varies from 0.254 to 0.367 for $-0.904 < m < 4$. The values of λ are given in Table 3.4.

β	1.6	1	0	- 0.199
λ	0.367	0.353	0.327	0.254

TABLE 3.4

H. Schuh[22], Leveque[38] and Lighthill[6] have found a solution for the equation 3.2.2 by assuming that the thermal boundary layer is very small compared to the hydrodynamic boundary layer and hence by replacing velocity profile by a tangent at the wall $f' = b\xi$. Thus an asymptotic solution of equation 3.2.2 is

$$\theta = 2^{-\frac{1}{3}} \, \Gamma(\tfrac{2}{3}) \, L \left[B_{-\frac{1}{3}}(L^3) - B_{\frac{1}{3}}(L^3)\right] \qquad (3.3.8)$$

where

$$L = \frac{2^{\frac{1}{3}}}{3^{\frac{1}{3}}} \left[Pr\, b\,(2-\beta)\right]^{1/6} \left[r^{\frac{1}{3}} \xi\right]^{\frac{1}{2}} \qquad (3.3.9)$$

and B_c the modified Bessel function of the first kind and order c, as defined, for instance, by McLachlan[31] and Γ the Gamma function. The heat transfer coefficient becomes,

$$\frac{Nu}{\sqrt{Re}} = 0.72895 \, \frac{(Pr\, b\, r)^{\frac{1}{3}}}{(2-\beta)^{1/6}} \qquad (3.4.0)$$

4. Description of Methods

4.1. Introduction

All the approximate methods, instead of satisfying the differential equation for every fluid particle, satisfy boundary conditions near the wall and the region of transition to the external flow, together with certain compatibility conditions. In the remaining region of the fluid in the boundary layer, only an average value over the differential equation is satisfied, the average being taken over the whole thickness of the boundary layer. Such mean values can be obtained from the momentum theory and the law of the conservation of energy. These are, in turn, derived from the equations of motion and energy by integrating over the boundary layer thickness. These integral equations of motion and heat flux, with assumed velocity and temperature profiles in the boundary layer, form the basis of the approximate methods.

All methods can be divided into three groups, depending on their applicability.

4.2. Arbitrary main stream velocity and uniform surface temperature

(a) Frossling

Frossling[29] solved the energy equation in a very similar manner to that of Pohlhausen[13] by assuming a power series for the main stream velocity, velocity in the boundary layer, and the temperature distribution in the boundary layer. This method is very cumbersome, particularly for slender body forms, where a large number of terms in the power expansion are required.

(b) Eckert

Eckert[1] solved the differential equation for the thickness of the thermal boundary layer by assuming that the temperature profile and the gradient of the thermal boundary layer thickness are the same, as on the wedge with the same velocity gradient outside the boundary layer. This method, although less than Frossling, involves lengthy and tedious calculations.

(c) Allen and Look

Allen and Look[20] have used Reynolds' analogy to obtain the heat transfer from the wall shear stress. The shear stress is obtained by assuming a velocity profile in the boundary layer similar to that over the flat plate. Frick and McCullough[21] extended this method for any Prandtl Number by suggesting a simple multiplier. This method, though very simple, gives very high values of heat transfer when compared to Eckert's exact solution for wedge flow.

Procedure for the calculation of heat transfer can be summarised as:

1. The calculation of the boundary layer thickness from

$$\delta^2 = \frac{5.3\nu\int_0^x u_1^{\,8.17}\,dx}{(u_1)^{9.17}} \tag{4.1}$$

The boundary layer thickness at the stagnation point is obtained from

$$\delta^2_{\text{Stag}} = \frac{\nu \bar{r}}{5u_1} \tag{4.2}$$

where $\bar{r}$ is the radius of curvature at the stagnation point.

2. The calculation of heat transfer coefficient from

$$\frac{hx}{k} = \lambda \left(\frac{x}{\delta}\right) (Pr)^{\frac{1}{3}} \tag{4.3}$$

where λ is the shape factor and is equal to 0.765 for the Blasius velocity profile. λ is assumed to be uniform up to the minimum pressure point and then reducing linearly to zero at the separation.

(d) Squire

Squire[4] assumed a Blasius velocity profile in the boundary layer and allowed the displacement thickness to vary with x in a manner appropriate to the assumed main stream velocity. He then solved the energy equation by assuming a temperature profile in the boundary layer similar to the velocity profile. This method contains more calculations in comparison to that of Smith and Spalding[9], Allen and Look[20] and Merk[37]. The method of Squire requires the following four steps.

1. Calculation of $\left(\frac{\Delta_1}{\delta_1}\right)^2 \phi\left(\frac{\Delta_1}{\delta_1}\right)$ from equation 4.4

$$\left(\frac{\Delta_1}{\delta_1}\right)^2 \phi\left(\frac{\Delta_1}{\delta_1}\right) = \frac{0.3861}{Pr} \frac{u_1^4 \int_0^x u_1 \, dx}{\int_0^x u_1^5 \, dx} \tag{4.4}$$

2. Determination of the values of $\left(\frac{\Delta_1}{\delta_1}\right)$ from the graph or table given by Squire. They are reproduced in Fig. 4.1 and Table 4.1 respectively.

3. Calculation of δ_1 from equation 4.5

$$\delta_1 = \frac{1.721}{u_1^3} \left[\nu \int_0^x u_1^5 \, dx \right]^{\frac{1}{2}} \tag{4.5}$$

4. Heat transfer from 2, 3 and equation 4.6

$$\frac{h}{k} = \frac{1}{\Delta_4} = \frac{0.5715}{\Delta_1} \tag{4.6}$$

$\frac{\Delta_1}{\delta_1}$	$\phi\left(\frac{\Delta_1}{\delta_1}\right)$	$\left(\frac{\Delta_1}{\delta_1}\right)^2 \phi\left(\frac{\Delta_1}{\delta_1}\right)$
0.5	0.2075	0.052
0.625	0.257	0.100
0.667	0.272	0.121
0.833	0.332	0.230
1.0	0.3861	0.386
1.25	0.4563	0.713
1.429	0.4988	1.018
1.667	0.5478	1.522
1.818	0.5750	1.901
2.0	0.5994	2.398

TABLE 4.1

(e) Smith and Spalding

Smith and Spalding(19) used the same procedure as that of Eckert(1) but replaced Eckert's graphical integration by a simple quadrature. Further, an error function is included to account for the deviation of the simple quadrature from the exact wedge solution. This method is very simple and still gives the results that are in agreement with Eckert. The equation derived by Smith and Spalding can be used only for fluids with Prandtl Number : 0.7. Smith and Shah(39, 42) have extended this method so that the heat transfer coefficient can be calculated for fluids with Prandtl Number in the range of $0.5 < Pr < 20{,}000$. This method reduces to the evaluation of Δ_4 from equation 4.7.

$$\frac{\Delta_4^2}{\nu} = \frac{A}{u_1^B} \int_0^x u_1^{B-1}\, dx + \frac{1}{u_1^B} \int_0^x u_1^{B-1}\, E_4\, dx \qquad (4.7)$$

and at the front stagnation point

$$\frac{\Delta_4^2}{\nu} = \frac{A/B}{\frac{d(u_1)}{dx}} \qquad (4.8)$$

where A and B are the constants depending on the Prandtl Number. Δ_4 is related to the heat transfer coefficient by the equation:

$$h = k/\Delta_4 \qquad (4.9)$$

The values of A and B for different Prandtl Numbers are presented in Table 4.2 and Fig. 4.2.

Pr	$A(Pr)^{\frac{2}{3}}$	B	$\frac{A}{B}(Pr)^{\frac{2}{3}}$
0.5	9.369	2.791	3.357
0.7	9.204	2.869	3.207
0.8	9.146	2.900	3.154
1.0	9.069	2.952	3.072
1.4	8.975	3.026	2.966
2	8.902	3.102	2.870
3	8.840	3.181	2.779
5	8.791	3.273	2.686
10	8.754	3.380	2.590
50	8.724	3.558	2.452
100	8.712	3.606	2.416
500	8.717	3.692	2.361
1000	8.716	3.716	2.346
5000	8.716	3753	2.322
10000	8.716	3.764	2.316
20000	8.716	3.773	2.310

TABLE 4.2

E4, which is a function of $\frac{\Delta_4^2}{\nu} \frac{du_1}{dx}$, is given in Ref. 39 for 0.7 < Pr < 10 and they are reproduced in Table 4.3. For Pr values greater than 10 it will often be satisfactory, when it is desired to include the error term in the calculation, to assume similarity of the E4 values with those for Pr : 10. By this it is meant that

$$\frac{E_4}{A} = f\left[\left(\frac{\Delta_4^2}{\nu} \frac{du_1}{dx}\right) \Big/ D\right] \tag{4.10}$$

where D is the interval of $\frac{\Delta_4^2}{\nu} \frac{du_1}{dx}$ between the values at which E_4 : 0.

Pr : 0.7	$\frac{\Delta_4^2}{\nu}\frac{du_1}{dx}$	6.05	4.07	2.26	1.01	0	-1.02
	E_4	1.042	0	-0.68	-0.67	0	2.06
Pr : 0.8	$\frac{\Delta_4^2}{\nu}\frac{du_1}{dx}$	5.427	3.663	2.039	0.917	0	-0.937
	E_4	1.027	0.011	-0.618	-0.614	0	1.935
Pr : 1.0	$\frac{\Delta_4^2}{\nu}\frac{du_1}{dx}$	4.541	3.074	1.721	0.779	0	-0.809
	E_4	0.919	-0.002	-0.551	0.543	0	1.716
Pr : 5	$\frac{\Delta_4^2}{\nu}\frac{du_1}{dx}$	1.327	0.919	0.531	0.248	0	-0.287
	E_4	0.343	0	-0.202	-0.208	0	0.740
Pr : 10	$\frac{\Delta_4^2}{\nu}\frac{du_1}{dx}$	0.798	0.554	0.325	0.154	0	-0.185
	E_4	0.238	0	-0.126	-0.132	0	0.491

TABLE 4.3

The error term E_4 is small and may be ignored when high accuracy is not demanded. If the error term is not excluded in equation 4.7, then it can be solved by an iteration procedure.

The experimental results of Shah[41] and Sogin et. al[37] have confirmed that this method predicts the heat transfer coefficient within 5%, except near separation, where it predicts rather high values.

Smith and Spalding[19] have also suggested another method in which they give an equation for mixed temperature thickness.

$$\frac{\Delta_m^2}{\nu} = \frac{19.78}{u_1^2} \int_0^x u_1 dx \tag{4.11}$$

where Δ_m is related to the heat transfer coefficient by

$$h = \frac{k\,C_m}{\Delta_m} \tag{4.12}$$

and C_m depends on $\left[\frac{\Delta_m^2}{\nu}\ \frac{du_1}{dx}\right]$. The graph and table for this relationship are reproduced in Fig. 4.3 and Table 4.4 respectively.

$\frac{\Delta_m^2}{\nu} \frac{du_1}{dx}$	9.89	4.88	1.96	0	-1.48
C_m	1.56	1.47	1.39	1.30	1.20

TABLE 4.4

This method has not been extended for a range of Prandtl Numbers. The equation 4.11 is, therefore, valid only for fluid with Prandtl Number : 0.7.

(f) Merk

Merk(36) has used the exact solution of Eckert(1) in deriving his approximate method, where he uses the wedge solutions of the dynamic problem in a mathematical argument leading to a series expansion in which only the first term is retained. This method is also simple and rapid as the method of Smith and Spalding. The heat transfer coefficient can be calculated from

$$\frac{\Delta_4}{\sqrt{\nu}} = \frac{\left[2\int_o^x u_1\, dx\right]^{\frac{1}{2}}}{u_1\,(E_o \ldots)} \tag{4.13}$$

where Δ_4 is related to the heat transfer coefficient by

$$\Delta_4 = \frac{k}{h} \tag{4.14}$$

The quantity E_o is a function of the Prandtl Number and the wedge variable Λ, which plays a role quite analogous to β and, indeed, is identically β for a pure wedge flow. The values of E_o, as a function of Λ for some values of the Prandtl Number, are reproduced in Fig. 4.4.

$$\Lambda = \left[\frac{2}{u_1^2}\int u_1\, dx\right]\frac{du_1}{dx} \tag{4.15}$$

The two dimensional stagnation value corresponds to Λ : 1, and the separation values to Λ : -0.1988.

4.3. Uniform main stream velocity with non-isothermal surface

4.3.1. Arbitrary surface temperature

The method for determining heat transfer for non-isothermal surfaces is similar to the methods used in determining the deflection of beams subjected to arbitrary load distributions. The energy equation of the boundary layer is linear in the fluid temperature, if the fluid properties are assumed to be constant. This allows the super-position technique to be employed. Rubesin has shown that the heat transfer rate for an arbitrary temperature variation can be determined by superimposing a number of "step temperature distribution", so that the summation of the steps yields the actual variable temperature distribution, and the heat transfer at any point is equal to the sum of the heat transfer rates attributed to all steps upstream of the point in question. This results in heat transfer rates from the non-isothermal given by the following integral expression.

$$\dot{q}''_{w(x)} = \int_{\xi=o}^{\xi=x} h(\xi\, x)\, d\, T_{w(\xi)} \tag{4.16}$$

Here the Kernel function $h(\xi\, x)$ is the heat transfer rate at position x due to step temperature rise $dT_{w(\xi)}$ at the position ξ. It should be noted that the integral of equation 4.16 must be taken in the "Stieltjes" sense rather than in the ordinary "Riemann" or "area" sense. Various investigators have obtained an equation of heat transfer rate for step rise in the surface temperature case, by assuming velocity and temperature profiles and then solving the energy equation.

Leveque[38] assumed a linear velocity profile, independent of x, and solved the differential energy equation. He obtained

$$h_{(\xi, x)} = \frac{k\, Pr^{\frac{1}{3}}}{3(\frac{1}{3})!} \left(\frac{\rho}{q\,\mu}\right)^{\frac{1}{3}} \left[\left(\frac{du}{dy}\right)_{y=0}\right]^{\frac{1}{3}} (x-\xi)^{-\frac{1}{3}} \tag{4.17}$$

Rubesin[18] assumed linear velocity and temperature profiles and solved the integral energy equation by further assuming that the thermal boundary layer varies proportionally to the momentum thickness. He obtained

$$h_{(\xi, x)} = \frac{0.304k}{x}\, Pr^{\frac{1}{3}}\, Re_x^{\frac{1}{2}} \left[1-(\xi/x)^{\frac{3}{4}}\right]^{-\frac{1}{3}} \tag{4.18}$$

Eckert[17] assumed cubic velocity and temperature profiles and solved the integral energy equation. He obtained

$$h_{(\xi, x)} = \frac{0.33k}{x}\, Pr^{\frac{1}{3}}\, Re_x^{\frac{1}{2}} \left[1-(\xi/x)^{\frac{3}{4}}\right]^{-\frac{1}{3}} \tag{4.19}$$

4.3.2. Arbitrary surface heat flux

Similar to a problem of finding heat transfer for an arbitrary surface temperature case, it is also an equally important problem to find surface temperature when the surface heat flux varies arbitrarily. Again, in a similar way, the wall temperature can be determined from

$$(T_w - T_1) = \int_{\xi=0}^{\xi=x} g(\xi, x)\, \dot{q}''_{w(\xi)}\, d\xi \tag{4.20}$$

where $g(\xi, x)$ is the wall temperature at x due to a unit heat flux at ξ .

Klein and Tribus[30] have shown a mathematical manipulation procedure by which the value of function $g(\xi, x)$ can be obtained from the known function $h(\xi, x)$.

Thus the values of the function $g(\xi, x)$ obtained from the equations of Leveque[38], Rubesin[18] and Eckert[17] are

$$g(\xi, x) = \frac{2}{3k} \frac{Pr^{-\frac{1}{3}}}{(\frac{2}{3})!} \left(\frac{\rho}{q\mu}\right)^{-\frac{1}{3}} \left[\left(\frac{du}{dy}\right)_{y=0}\right]^{-\frac{1}{3}} (x - \xi)^{-\frac{2}{3}} \tag{4.21}$$

$$g(\xi, x) = \frac{1}{6(\frac{1}{3})!\ (\frac{2}{3})!} \times \frac{Pr^{-\frac{1}{3}}}{0.304k} Re_x^{-\frac{1}{2}} \left[1 - (\xi/x)^{\frac{3}{4}}\right]^{-\frac{2}{3}} \tag{4.22}$$

and

$$g(\xi, x) = \frac{1}{6(\frac{1}{3})!\ (\frac{2}{3})!} \frac{Pr^{-\frac{1}{3}}}{0.33k} Re_x^{-\frac{1}{2}} \left[1 - (\xi/x)^{\frac{3}{4}}\right]^{-\frac{2}{3}} \tag{4.23}$$

respectively.

Smith and Shah[40] assumed the cubic velocity and temperature profiles and obtained a very simple equation for an arbitrary heat flux problem by directly integrating the energy equation for a step heat flux case. They obtained

$$(T_w - T_1) = \frac{2.395\, (Re_x)^{\frac{1}{2}}\, (Pr)^{\frac{2}{3}}}{\rho\, u_1\, C_p} \int_0^x (1 - \xi/x)^{\frac{1}{3}}\, d\, \dot{q}''_w(\xi) \tag{4.24}$$

4.4. Arbitrary main stream velocity and arbitrary surface temperature

(a) Lighthill

Lighthill[6] solved the energy equation in Von Mises form, assuming a linear velocity profile in the thermal boundary layer. He obtained

$$h(\xi, x) = \frac{k}{(\frac{1}{3})!} Pr^{\frac{1}{3}} \left(\frac{\rho}{q\mu^2}\right)^{\frac{1}{3}} \sqrt{\tau_{\omega(x)}} \left[\int_{x_0}^{x} \sqrt{\tau_{\omega(z)}}\, dz\right]^{-\frac{1}{3}} \tag{4.25}$$

and therefore for an arbitrary surface temperature case

$$\frac{Nu}{\sqrt{Re}} = \frac{0.486\ Pr^{\frac{1}{3}}}{(T_w - T_1)_x} \sqrt{\frac{x}{u_1}} \times \frac{1}{(\mu\rho)^{1/6}\sqrt{\tau_{\omega(x)}}} \int_0^x \left[\int_{\xi=0}^{\xi=x} \sqrt{\tau_{\omega(z)}}\, dz \right]^{-\frac{1}{3}} dT_{w(\xi)} \qquad \ldots\ldots \quad (4.26)$$

For wedge flow

$$\tau_\omega = \mu \frac{u_1}{x} \sqrt{\left(\frac{m+1}{2}\right)\left(\frac{u_1 x}{\nu}\right)} \left[f''_o \right] \qquad (4.27)$$

and the heat transfer coefficient for wedge flow is given by

$$\frac{Nu}{\sqrt{Re}} = \frac{0.3935}{(T_w - T_1)} Pr^{\frac{1}{3}} (m+1)^{\frac{1}{2}} f''_{(o)} \int_{\xi=o}^{\xi=x} \left[1 - (\xi/x)^{\frac{3}{4}(m+1)} \right]^{-\frac{1}{3}} dT_{w(\xi)} \qquad (4.28)$$

Using a similar approach to that of Klein and Tribus[30], the equation for an arbitrary heat flux becomes:

$$g_{(\xi,x)} = \frac{2}{9(\frac{2}{3})!\ k} x \left(\frac{9\mu^2}{\rho Pr}\right)^{\frac{1}{3}} \frac{1}{\sqrt{\tau_\omega}} \left[\int_{x_o}^{x} \tau_{\omega(z)}\, dz \right]^{-\frac{2}{3}} \qquad (4.29)$$

This method is expected to give good results for the fluid with high Prandtl Number, as the approximation of a tangential velocity profile at the wall is true only for the case when the temperature boundary layer is a small fraction of the velocity boundary layer. However, it is found to give fairly good results even when the Prandtl Number is as low as 0.7. This method needs an initial knowledge of the distribution of the shear stress over the surface.

(b) Bond

Bond[43] also assumed a linear velocity profile and solved the differential equation for wedge flow. For the wedge flow be obtained

$$h_{(\xi,x)} = \left[\frac{1+m}{2}\right]^{\frac{1}{2}} \frac{k}{(\frac{1}{3})!\ x}\ b^{\frac{1}{3}}\ Re_x^{\frac{1}{2}} \left[1 - (\xi/x)^c \right]^{-\frac{1}{3}} \qquad (4.30)$$

and

$$g_{(\xi,x)} = \frac{2c}{9(\frac{2}{3})!\ k} \left(\frac{2}{1+m}\right)^{\frac{1}{2}} b^{-\frac{1}{3}}\ Re_x^{-\frac{1}{2}} \left(\frac{x}{\xi}\right)^{\frac{m+1}{2}} \left(\frac{x^c - \xi^c}{\xi^c}\right)^{-\frac{2}{3}} \qquad (4.31)$$

where

$$\left.\begin{array}{l} b : Pr/6\ \ f''(o) \\ c : \frac{3}{4}(1+m) \end{array}\right\} \qquad (4.32)$$

and

(c) Ambrok

Ambrok[10] has derived an approximate method for calculating the heat transfer coefficient by assuming that a relation of the type $Nu : A(Re\ \Delta_2)^n$, which is valid for a flat plate with a uniform surface temperature, is also valid for a problem where main stream velocity and surface temperature vary arbitrarily. With this assumption he has solved an integral energy equation. He obtained

$$\frac{Nu}{\sqrt{Re}} = \frac{0.332\ Pr^{\frac{1}{3}}\ (T_w - T_1)}{\left[\frac{1}{u_1 x}\int_{x=x_o}^{x} u_1 (T_w - T_1)^2\ dx\right]^{\frac{1}{2}}} \qquad (4.33)$$

where heating starts at $x : x_o$. This method is very simple but it is found to give low values of heat transfer when compared to those of Eckert for wedge flow.

(d) Spalding

Spalding[12] has improved on Lighthill's method by a correction which accounts for the departure from linearity of the velocity profile within the thermal boundary layer, and which comprehends the influences of Prandtl Number, pressure gradient, body forces and non-coincident start of velocity and thermal layers.

In this method, the momentum thickness, δ_2, of the velocity boundary layer is to be evaluated first by a procedure similar to those of Walz[35] and Thwaites. This involves the evaluation of

$$\frac{\delta_2^2}{\nu} = \frac{0.4418}{u_1^{5.17}} \int_o^x u_1^{4.17}\ dx - \frac{1}{u_1^{5.17}} \int_o^x u_1^{4.17}\ E_2 dx \qquad (4.34)$$

where E_2 is a tabulated function of $\frac{\delta^2}{\nu}\ \frac{du_1}{dx}$

$\frac{\delta_2^2}{\nu}\ \frac{du_1}{dx}$	E_2	$\left(\frac{\delta_4}{\delta_2}\right)^2$
-0.0682	-0.026	∞
0.0266	-0.0058	36.9
0	0	20.5
0.0333	0.0033	12.75
0.0611	0.0019	9.50
0.0855	0.0	7.70

TABLE 4.5

The integral containing E_2 is once again a small correction term which can be omitted for most purposes.

With $\delta_{2}(x)$ given by equation 4.34, $\delta_{4}(x)$ is obtained by interpolation from a table of the ratio δ_4/δ_2 with argument $\left[\frac{\delta_2^2}{\nu}\frac{du_1}{dx}\right]$. δ_4 is the "shear thickness".

Thereafter, the heat transfer coefficient can be obtained by the evolution of the equation 4.35

$$\Delta_4 \quad \left(\frac{\delta_4}{u_1}\right)^{\frac{1}{2}} \left[6.41\,\alpha \int_{x_o}^{x} \left(\frac{u_1}{\delta_4}\right)^{\frac{1}{2}} dx + \alpha \int_{x_o}^{x} \left(\frac{u_1}{\delta_4}\right)^{\frac{1}{2}} F\,dx \right]^{\frac{1}{3}} \tag{4.35}$$

where F is a graphically presented function of the argument $\frac{\delta_4 \Delta_4}{\nu}\frac{du_1}{dx}$ (Fig. 4.5), and x_o is the value of x where the heating starts.

This time the correction term F is not always small. Since Δ_4 appears in its argument, some iteration is necessary.

(e) <u>Schuh</u>

Schuh(5) selected a dependent variable, which is a function of the ratio of heat flow across the whole boundary layer to the temperature gradient at the wall in a suitable dimensionless form, and then integrated the momentum and the energy equations.

Schuh's method involves similar steps to those of Spalding's method, namely:

1. The determination of δ_2 by the Walz-Thwaites technique.
2. Calculation of the auxiliary function $\frac{\delta_2^2}{\nu}\frac{du_1}{dx}$ from 1. and the known velocity gradient.
3. β and Z_1 are the functions of $\frac{\delta_2^2}{\nu}\frac{du_1}{dx}$ and to be read from the graph (Fig. 4.6).
4. Calculation of n from equation 4.36

$$n = \left(\frac{x}{T_w - T_1}\right) \frac{dT_w}{dx} \tag{4.36}$$

5. Calculation of functions P and G from equations 4.37 and 4.38 by an iterative procedure.

$$P = \left[\frac{3}{2Pr}\right]^{\frac{1}{3}} A^{1/6} \frac{1}{\left[\left[T_W - T_1\right] G\,(2-\beta)\right]^{\frac{1}{2}} \left[u_1 x\right]^{\frac{1}{4}}}$$

$$\times \left[\int_{x_o}^{x} \left[\frac{u_1^3}{(2-\beta)x}\right]^{\frac{1}{4}} \frac{G^{3/2}\,(T_W - T_1)^{3/2}}{A^{\frac{1}{2}}}\,dx\right]^{\frac{1}{3}} \tag{4.37}$$

and

$$G : 0.57\,(\beta + 0.205)^{0.104}\,A^{0.37+0.06\beta}\,(2-\beta)^{\frac{1}{2}}\,P^{1-3\tau} \tag{4.38}$$

where

$$A : 1 + (2-\beta)n \tag{4.39}$$

and values of τ are given in Table 4.6.

β	1.6	1.0	0	-0.199
τ	0.367	0.355	0.327	0.254

TABLE 4.6

Equation 4.38 is to be used for $n \leqslant 4$ and $0.3 \leqslant P \leqslant 1.3$ but for $n \geqslant 4$, $-0.14 \leqslant \beta \leqslant 1.0$ and $P \geqslant 0.2$, the value of function G to be obtained from equation 4.40 and Table 4.7

$$G = C\left(\frac{\delta_2^2}{\nu}\,\frac{du_1}{dx}\right)\,n^{\frac{1}{3}} \tag{4.40}$$

6. Heat transfer coefficient then to be calculated from equation 4.41

$$\frac{\Delta_4}{\sqrt{\nu}} = \frac{P}{G}\sqrt{\frac{x}{u_1}}\sqrt{2-\beta} \tag{4.41}$$

β	$\frac{\delta_2^2}{\nu}\,\frac{du_1}{dx}$	$C\left[\frac{\delta_2^2}{\nu}\,\frac{du_1}{dx}\right]$
-0.14	-0.041	0.583
0	0	0.714
0.2	0.033	0.782
0.4	0.053	0.809
0.6	0.068	0.814
0.8	0.078	0.805
1.0	0.0854	0.781

TABLE 4.7

(f) Seban and Drake

Seban[2] and Drake[3] solved the momentum equation by a procedure similar to that of Eckert, and obtained heat transfer rate by assuming that it is related to the momentum thickness in the same manner as on the wedge.

This method also involves similar steps to those of Spalding and Schuh.

1. The determination of δ_2 and the auxiliary function $\frac{\delta_2^2}{\nu} \frac{du_1}{dx}$.

2. Evaluation of function Z_1 and β from Fig. 4.6.

3. The rate of heat transfer at the wall for a particular case, when wall temperature is a power function of length. e.g. $(T_w - T_1) : AX^n$, is given by

$$\dot{q}''_w = -k A x^n g'(o) \frac{Z_1}{\delta_2} \qquad (4.42)$$

or

$$\frac{\Delta_4}{\delta_2} = \frac{1}{g'(o) Z_1} \qquad (4.43)$$

Where $g'(o)$ is a function of β, n and Pr. The value of $g'(o)$ is relatively insensitive to β but depends critically upon the exponent n of the wall temperature function Ax^n. The function $g'(o)$ for Prandtl Number : 0.7 is shown in Fig. 4.7.

By superimposing, the heat transfer coefficient can be determined for a variation of the surface temperature of the type

$$(T_w - T_1) = A_1 x^{n_1} + A_2 x^{n_2} + A_3 x^{n_3} \qquad (4.44)$$

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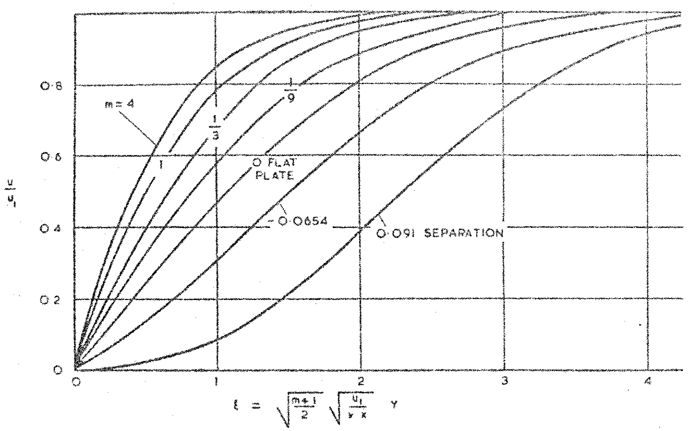

FIG. 3·1. VELOCITY DISTRIBUTION IN THE LAMINAR BOUNDARY LAYER IN THE FLOW PAST A WEDGE.

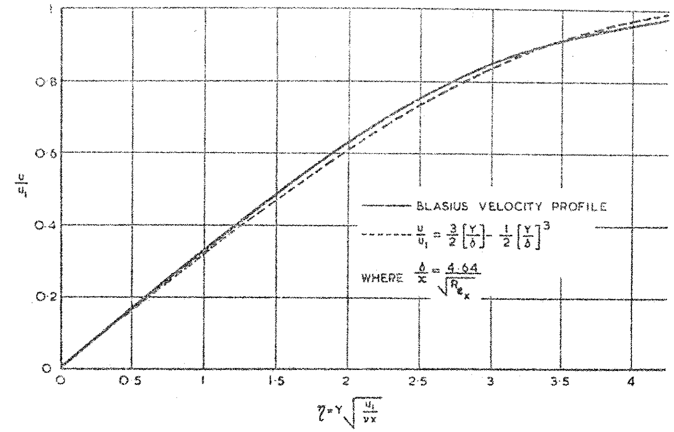

FIG. 3·2. VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ALONG A FLAT PLATE.

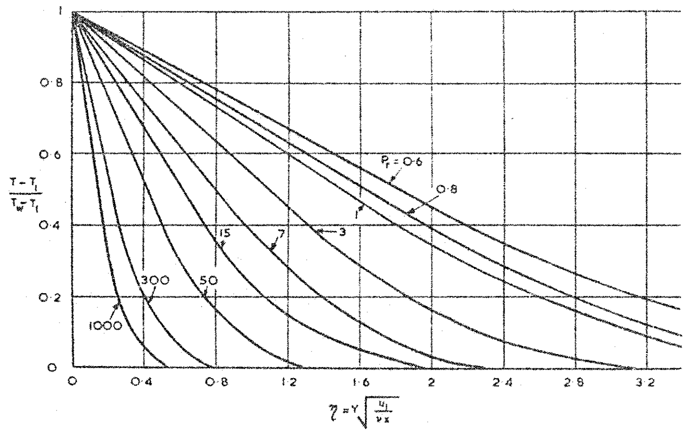

FIG. 3·3. TEMPERATURE DISTRIBUTION OVER HEATED PLATE FOR VARIOUS PRANDTL NUMBERS.

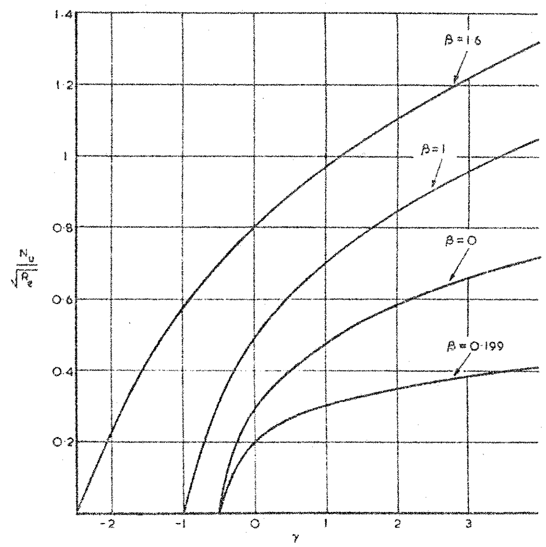

FIG. 3·4. LOCAL HEAT TRANSFER COEFFICIENTS FOR WEDGE FLOWS WITH VARIABLE WALL TEMPERATURE PRANDTL NUMBER = 0·7.

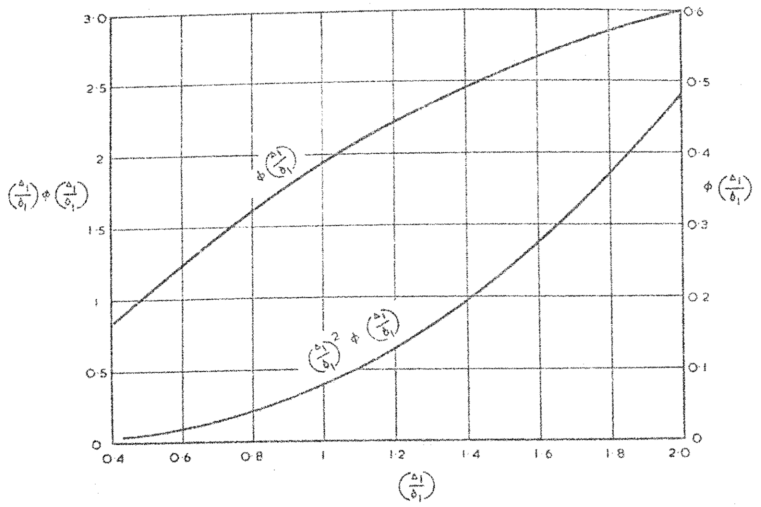

FIG. 4·1. GRAPH OF THE FUNCTIONS $\left(\frac{\Delta_1}{\delta_1}\right)^2 \phi\left(\frac{\Delta_1}{\delta_1}\right)$ AND $\phi\left(\frac{\Delta_1}{\delta_1}\right)$.

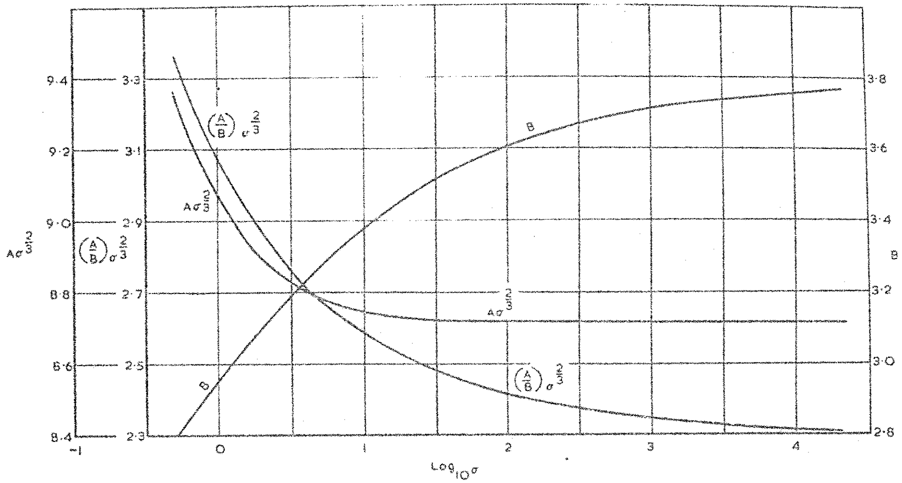

FIG. 4·2. GRAPHS OF DEPENDENCE OF NUMBERS A AND B ON σ.

σ = PRANDTL NUMBER

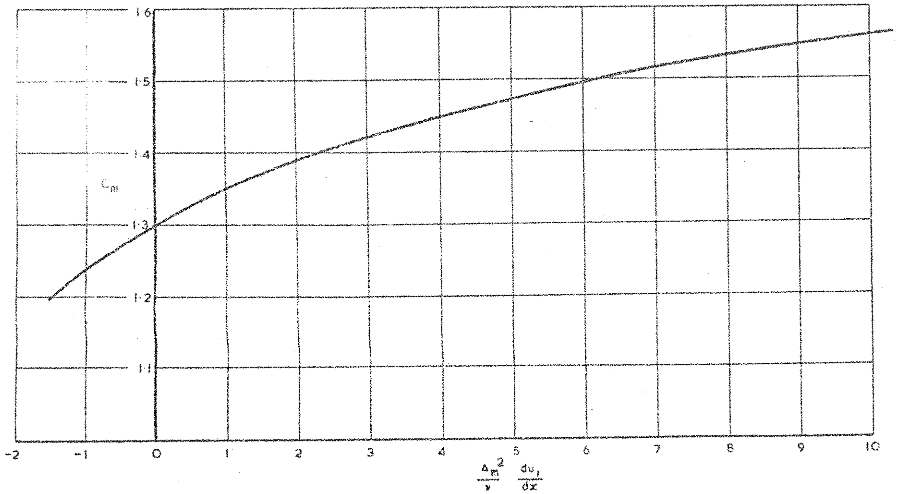

FIG. 4.3. C_m AS A FUNCTION OF $\frac{\Delta_m{}^2}{\nu} \frac{du_1}{dx}$.

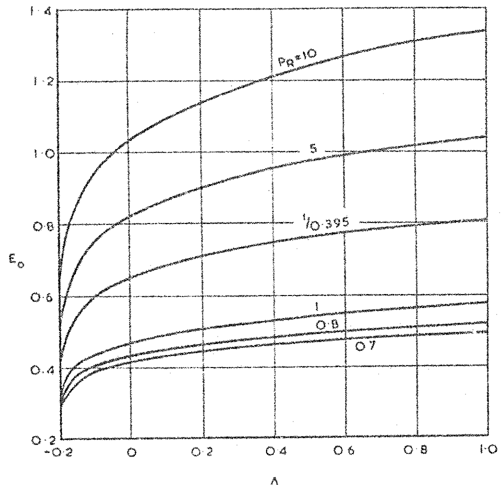

FIG. 4·4. E_o AS A FUNCTION OF THE WEDGE VARIABLE Λ, FOR SOME VALUES OF THE PRANDTL NUMBER.

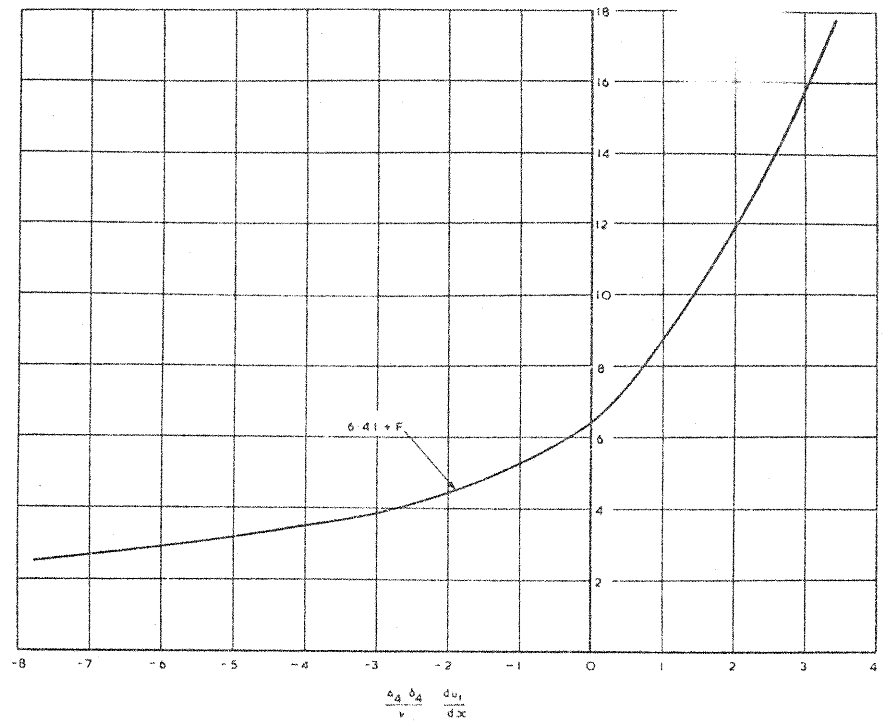

FIG. 4·5. GRAPH OF THE FUNCTION $F\left(\frac{\Delta_4\, \delta_4}{\nu} \quad \frac{du_1}{dx}\right)$.

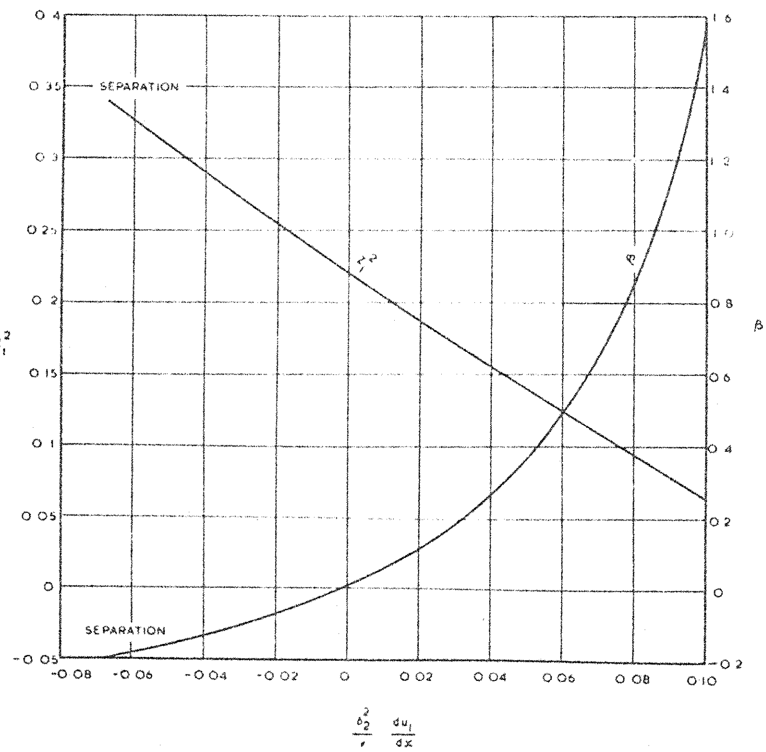

FIG. 4·6. HARTREE PARAMETER β AND z_1^2 AS A FUNCTION OF PARAMETER $\frac{\delta_2^2}{\nu}\,\frac{du_1}{dx}$.

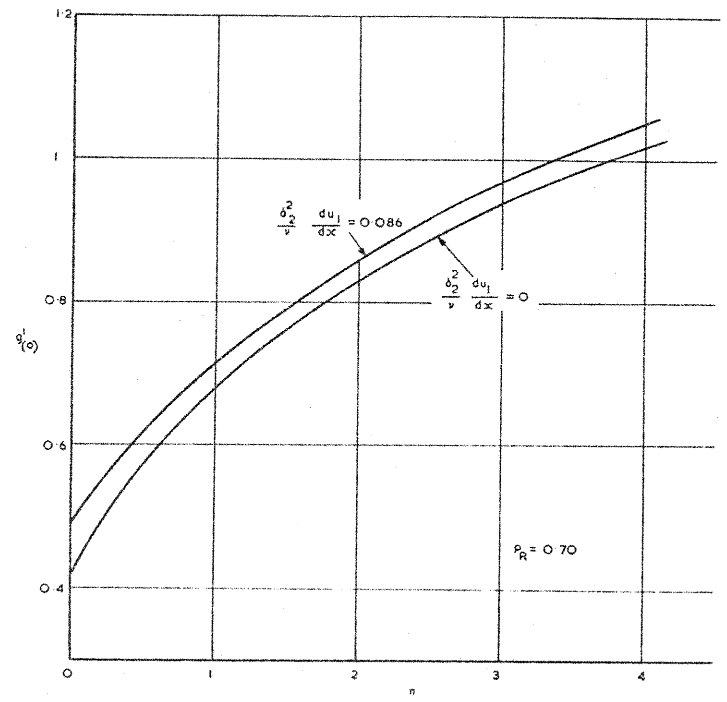

FIG. 4·7. SURFACE TEMPERATURE GRADIENT IN THERMAL BOUNDARY LAYER.